

RICCI FLOW WITH SURGERY IN HIGHER DIMENSIONS

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ABSTRACT. We present a new curvature condition which is preserved by the Ricci flow in dimension $n \geq 4$. For initial metrics satisfying this condition, we establish a higher dimensional version of Hamilton's neck-like curvature pinching estimate. Using this estimate, we are able to prove a version of Perelman's Canonical Neighborhood Theorem in higher dimensions. This makes it possible to extend the flow beyond singularities by a surgery procedure in the spirit of Hamilton and Perelman. As a corollary, we obtain a classification of all diffeomorphism types of such manifolds in terms of a connected sum decomposition. In particular, the underlying manifold cannot be an exotic sphere.

Our result is sharp in many interesting situations. For example, the curvature tensors of $\mathbb{CP}^{n/2}$, $\mathbb{HP}^{n/4}$, $S^{n-k} \times S^k$, $S^{n-2} \times \mathbb{H}^2$, $S^{n-2} \times \mathbb{R}^2$ all lie on the boundary of our curvature cone. Another borderline case is the pseudo-cylinder: this is a rotationally symmetric hypersurface which is weakly, but not strictly, two-convex. Finally, the curvature tensor of $S^{n-1} \times \mathbb{R}$ lies in the interior of our curvature cone.

1. INTRODUCTION

Since its introduction by Hamilton [13] in 1982, the Ricci flow has become a fundamental tool in Riemannian geometry. In particular, two lines of research have been pursued:

First, under what conditions does the Ricci flow deform a given initial metric to a metric of constant curvature? The earliest result in this direction is the famous work of convergence theorem of Hamilton for three-manifolds with positive Ricci curvature. In higher dimensions, Huisken [18], Margerin [21], and Nishikawa [25] found various pinching conditions that guarantee the convergence of the flow to a round metric. These conditions were later weakened in work of Margerin [22], [23]. In his fundamental work [14], Hamilton introduces his PDE-ODE principle, and used it showed that the Ricci flow deforms any metric with positive curvature operator in dimension 4 to a round metric. This was generalized to higher dimensions in an important paper by Böhm and Wilking [1]. Finally, in [6] it was shown that the Ricci flow converges to a round metric, provided that the initial metric satisfies the so-called PIC2 condition. This curvature condition, which is equivalent to positive complex sectional curvature, is implied by 1/4-pinching. It follows that the flow evolves any 1/4-pinned metric to a round metric, thereby proving the Differentiable Sphere Theorem.

Second, it is of interest to find conditions that restrict the singularities that can form under the evolution to so-called neck pinch singularities. This was first done in an influential paper by Hamilton [17], where he used the Ricci flow to classify four-manifolds with positive isotropic curvature. In a striking breakthrough, Perelman [26],[27],[28] succeeded in carrying out a similar program in dimension 3, without any assumptions on the initial metric, proving the Poincaré conjecture as a direct consequence.

Our goal in this paper is to prove the following higher-dimensional version of Hamilton's theorem [17]:

Theorem 1.1. *Let (M, g_0) be a compact n -dimensional manifold whose curvature tensor lies in the interior of the cone*

$$\{R = S + H \otimes \text{id} : S \in \text{PIC2}, \text{Ric}_0(S) = 0, \\ \text{tr}(H) \text{id} - (n - 4) H \geq 0\}$$

at each point. Moreover, suppose that M does not contain non-trivial incompressible space forms S^{n-1}/Γ . Then there exists a Ricci flow with surgery starting from (M, g_0) . This flow involves performing finitely many surgeries on necks of the form $S^{n-1} \times I$. Moreover, the flow becomes extinct in finite time.

Theorem 1.2. *Under the assumptions of Theorem 1.1, M is diffeomorphic to a connected sum of finitely many pieces. Each piece is a quotient of S^n or a compact quotient of $S^{n-1} \times \mathbb{R}$.*

Remark 1.3. • The curvature tensors of $\mathbb{CP}^{n/2}$, $\mathbb{HP}^{n/4}$, $S^{n-k} \times S^k$, $S^{n-2} \times \mathbb{H}^2$, $S^{n-2} \times \mathbb{R}^2$ all lie on the boundary of our curvature cone.

- The set

$$\{H \otimes \text{id} : \text{tr}(H) \text{id} - (n - 4) H \geq 0\}$$

is the convex hull of the set of all curvature tensors of pseudo-cylinders. (By a pseudo-cylinder, we mean a rotationally symmetric hypersurface which is weakly, but not strictly, two-convex.)

- The curvature tensor of $S^{n-1} \times \mathbb{R}$ lies in the interior of our curvature cone. Consequently, our curvature condition is preserved under formation of connected sums.
- For $n = 4$, our curvature condition reduces to $\min\{a_1, c_1\} \geq 0$ (in the notation of [14] and [17]).

The assumption that M does not contain non-trivial incompressible space forms can be removed by working with orbifolds instead of smooth manifolds. This was done in dimension 4 by Chen-Tang-Zhu [10]. Similar adaptations work in higher dimensions (cf. [10], p. 47). This yields the following result:

Theorem 1.4. *Let (M, g_0) be a compact n -dimensional orbifold whose curvature tensor lies in the interior of the cone*

$$\{R = S + H \otimes \text{id} : S \in \text{PIC2}, \text{Ric}_0(S) = 0, \\ \text{tr}(H) \text{id} - (n - 4) H \geq 0\}$$

at each point. Then M is diffeomorphic to a connected sum of finitely many pieces which fall into one of the following categories:

- An orbifold of the form S^n/Γ , where Γ is a group of isometries of S^n (not necessarily acting freely).
- A compact quotient of $S^{n-1} \times \mathbb{R}$.
- An orbifold of the form $(B^n/\Gamma) \cup_f (B^n/\Gamma)$. Here, Γ is a group of isometries of S^{n-1} acting freely. Moreover, $f : S^{n-1}/\Gamma \rightarrow S^{n-1}/\Gamma$ is an isometry from the quotient S^{n-1}/Γ to itself.

In particular, if M is a smooth manifold satisfying the curvature condition in Theorem 1.4, then the fundamental group of M is virtually free (cf. [10]).

In this paper, we will focus on the proof of Theorem 1.1. The adaptation to the orbifold setting is similar to [10]; the details will appear elsewhere.

2. CURVATURE PINCHING ESTIMATES FOR THE RICCI FLOW IN HIGHER DIMENSIONS

We recall the following notation:

$$\begin{aligned} B(S, T)_{ijkl} = & \frac{1}{2} \sum_{p,q=1}^n (S_{ijpq} T_{klpq} + S_{klpq} T_{ijpq}) \\ & + \sum_{p,q=1}^n (S_{ipkq} T_{jplq} - S_{iplq} T_{jpkq} - S_{jpkq} T_{iplq} + S_{jplq} T_{ipkq}) \end{aligned}$$

Clearly, $B(S, T) = B(T, S)$ and $B(R, R) = Q(R)$, where $Q(R)$ is the term appearing in Hamilton's curvature ODE.

Moreover, we define

$$(S * H)_{ik} = \sum_{j,l=1}^n S_{ijkl} H_{jl}.$$

Finally, we denote by $(A \oslash B)_{ijkl} = A_{ik}B_{jl} - A_{il}B_{jk} - A_{jk}B_{il} + A_{jl}B_{ik}$ the Kulkarni-Nomizu product of two symmetric two-tensors A and B .

Lemma 2.1. *We have*

$$B(S, H \oslash \text{id}) = \text{Ric}(S) \oslash H + (S * H) \oslash \text{id}.$$

and

$$\begin{aligned} Q(H \oslash \text{id}) = & (n-2) H \oslash H + 2 \text{tr}(H) H \oslash \text{id} \\ & - 2 H^2 \oslash \text{id} + |H|^2 \text{id} \oslash \text{id}. \end{aligned}$$

Proof. We begin with the first statement. If we put $T = H \otimes \text{id}$, then we obtain

$$\begin{aligned} \frac{1}{2} \sum_{p,q=1}^n S_{ijpq} T_{klpq} &= \frac{1}{2} \sum_{p,q=1}^n S_{ijpq} (H_{kp} \delta_{lq} - H_{kq} \delta_{lp} - H_{lp} \delta_{kq} + H_{lq} \delta_{kp}) \\ &= \sum_{p=1}^n S_{ijpl} H_{kp} + \sum_{p=1}^n S_{ijkp} H_{lp} \end{aligned}$$

and

$$\begin{aligned} \sum_{p,q=1}^n S_{ipkq} T_{jplq} &= \sum_{p,q=1}^n S_{ipkq} (H_{jl} \delta_{pq} - H_{jq} \delta_{lp} - H_{lp} \delta_{jq} + H_{pq} \delta_{jl}) \\ &= \text{Ric}(S)_{ik} H_{jl} - \sum_{p=1}^n S_{ilkp} H_{jp} - \sum_{p=1}^n S_{ipkj} H_{lp} + (S * H)_{ik} \delta_{jl}. \end{aligned}$$

Putting these facts together, we obtain

$$\begin{aligned} B(S, T)_{ijkl} &- (\text{Ric}(S) \otimes H)_{ijkl} - ((S * H) \otimes \text{id})_{ijkl} \\ &= \sum_{p=1}^n S_{ijpl} H_{kp} + \sum_{p=1}^n S_{ijkp} H_{lp} \\ &+ \sum_{p=1}^n S_{pjkl} H_{ip} + \sum_{p=1}^n S_{ipkl} H_{jp} \\ &- \sum_{p=1}^n S_{ilkp} H_{jp} - \sum_{p=1}^n S_{ipkj} H_{lp} \\ &+ \sum_{p=1}^n S_{iklp} H_{jp} + \sum_{p=1}^n S_{iplj} H_{kp} \\ &+ \sum_{p=1}^n S_{jlkp} H_{ip} + \sum_{p=1}^n S_{jpkp} H_{lp} \\ &- \sum_{p=1}^n S_{jklp} H_{ip} - \sum_{p=1}^n S_{jpli} H_{kp}, \end{aligned}$$

and the right hand side vanishes by the first Bianchi identity. This proves the first statement.

To derive the second statement, we apply the first statement with $S = H \otimes \text{id}$. Then $\text{Ric}(S) = (n-2)H + \text{tr}(H)\text{id}$. Moreover, $S * H = \text{tr}(H)H - 2H^2 + |H|^2 \text{id}$. From this, the assertion follows.

Definition 2.2. Given $\sigma \in (0, 2]$ and $\theta \geq 0$, we define a cone $\mathcal{C}_{\sigma, \theta}$ in the space of algebraic curvature tensors by

$$\begin{aligned} \mathcal{C}_{\sigma, \theta} := \{ & R = S + H \otimes \text{id} : S \in \text{PIC2}, \text{Ric}_0(S) = 0, \\ & \text{tr}(H) \text{id} - (n - 2\sigma) H \geq 0, \\ & \text{tr}(H) - \theta \text{scal}(S) \geq 0\}. \end{aligned}$$

Equivalently, $\mathcal{C}_{\sigma, \theta}$ consists of all algebraic curvature tensors R with the property that

$$R - \frac{1}{n-2} \text{Ric}_0 \otimes \text{id} - \frac{1}{n} \frac{\theta}{1 + 2(n-1)\theta} \text{scal} \text{id} \otimes \text{id} \in \text{PIC2}$$

and

$$|v|^2 R - \frac{1}{n-2} |v|^2 \text{Ric}_0 \otimes \text{id} - \frac{1}{2} \frac{n-2\sigma}{n-2} \text{Ric}_0(v, v) \text{id} \otimes \text{id} \in \text{PIC2}$$

for each v .

We first examine the properties of the cones $\mathcal{C}_{1,0}$ and $\mathcal{C}_{2,0}$.

Proposition 2.3. *Suppose that $R \in \mathcal{C}_{1,0}$. Then $R \in \text{PIC2}$. Moreover, if $\text{Ric}(v, v) = 0$ for some unit vector v , then $R = c(\text{id} - 2v \otimes v) \otimes \text{id}$ for some $c \geq 0$. In other words, R is the curvature tensor of a cylinder $S^{n-1} \times \mathbb{R}$.*

Proof. By definition, we may write $R = S + H \otimes \text{id}$, where $S \in \text{PIC2}$, $\text{Ric}_0(S) = 0$, and $\text{tr}(H) \text{id} - (n-2)H \geq 0$. The condition $\text{tr}(H) \text{id} - (n-2)H \geq 0$ easily implies that H is two-nonnegative. Consequently, $H \otimes \text{id} \in \text{PIC2}$, and this implies $R \in \text{PIC2}$. Moreover, the Ricci tensor of R is given by $\frac{1}{n} \text{scal}(S) \text{id} + \text{tr}(H) \text{id} + (n-2)H$. The condition $\text{tr}(H) \text{id} - (n-2)H \geq 0$ implies $\text{tr}(H) \text{id} + (n-2)H \geq 0$. Hence, if $\text{Ric}(v, v) = 0$, then we have $\text{scal}(S) = 0$ and furthermore $H(v, v) = -\frac{1}{n-2} \text{tr}(H)$. From this, we deduce that $S = 0$ and $H = c(\text{id} - 2v \otimes v)$ for some $c \geq 0$.

Proposition 2.4. *The curvature tensor of $S^{n-1} \times \mathbb{R}$ lies in the interior of the cone $\mathcal{C}_{2,0}$. Moreover, the curvature tensors of $S^{n-2} \times \mathbb{R}^2$ and $S^{n-2} \times S^2$ lie in $\mathcal{C}_{2,0}$. Here, the curvature tensor of $S^{n-2} \times S^2$ is normalized so that the trace-free Ricci part vanishes. Finally, the curvature tensor of a pseudo-cylinder lies in $\mathcal{C}_{2,0}$.*

Proof. Let

$$S_{ijkl} = \begin{cases} (n-3)(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) & \text{if } i, j, k, l \in \{1, 2\} \\ \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} & \text{if } i, j, k, l \in \{3, \dots, n\} \\ 0 & \text{otherwise.} \end{cases}$$

Then $S \in PIC2$, and the trace-free Ricci part of S vanishes. Geometrically, S is the curvature tensor of $S^2 \times S^{n-2}$ (suitably normalized). Moreover, let

$$H_{ij} = \begin{cases} -\delta_{ij} & \text{if } i, j \in \{1, 2\} \\ \delta_{ij} & \text{if } i, j \in \{3, \dots, n\} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\text{tr}(H)\text{id} - (n-4)H \geq 0$. Geometrically, $H \oslash \text{id}$ represents the curvature tensor of $\mathbb{H}^2 \times S^{n-2}$ (suitably normalized). Finally, we observe that $S + \frac{n-3}{2}H \oslash \text{id}$ is the curvature tensor of $\mathbb{R}^2 \times S^{n-2}$. This shows that $S^2 \times S^{n-2}$ and $\mathbb{R}^2 \times S^{n-2}$ both lie in the cone $\mathcal{C}_{2,0}$. Finally, the fact that the pseudo-cylinder lies in $\mathcal{C}_{2,0}$ is clear from the definition.

We now state the main result of this section:

Theorem 2.5. *For each n , there exists a constant $\bar{\theta} = \bar{\theta}(n)$ with the following property: For each $\sigma \in (0, 2]$ and each $\theta \in [0, \bar{\theta}]$, the cone $\mathcal{C}_{\sigma, \theta}$ is invariant under the Hamilton ODE. Moreover, if $\sigma \in (0, 1) \cup (1, 2)$ and $\theta \in (0, \bar{\theta})$, the cone $\mathcal{C}_{\sigma, \theta}$ is transversally invariant away from 0.*

Proof. In the following, we fix real numbers $\sigma \in (0, 2]$ and $\theta \geq 0$. We evolve S and H by

$$\begin{aligned} \frac{d}{dt}S &= Q(S) + (n-2)H \oslash H - 2\text{tr}(H)H \oslash \text{id} + 2H^2 \oslash \text{id} \\ &\quad + \frac{2}{\sigma(n-2\sigma)}\text{tr}(H)^2\text{id} \oslash \text{id} - \frac{2-\sigma}{\sigma}|H|^2\text{id} \oslash \text{id} \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt}H &= 2S * H + \frac{2}{n}\text{scal}(S)H + 4\text{tr}(H)H - 4H^2 \\ &\quad - \frac{2}{\sigma(n-2\sigma)}\text{tr}(H)^2\text{id} + \frac{2}{\sigma}|H|^2\text{id}. \end{aligned}$$

We proceed in several steps:

Step 1: We claim that the condition $\text{Ric}_0(S) = 0$ is preserved. To show this, it suffices to prove that the term

$$\begin{aligned} T &:= (n-2)H \oslash H - 2\text{tr}(H)H \oslash \text{id} + 2H^2 \oslash \text{id} \\ &\quad + \frac{2}{\sigma(n-2\sigma)}\text{tr}(H)^2\text{id} \oslash \text{id} - \frac{2-\sigma}{\sigma}|H|^2\text{id} \oslash \text{id} \end{aligned}$$

has vanishing trace-free Ricci part.

The Ricci tensor of $(n-2)H \otimes H - 2\operatorname{tr}(H)H \otimes \operatorname{id} + 2H^2 \otimes \operatorname{id}$ is given by

$$\begin{aligned} & (n-2)[2\operatorname{tr}(H)H - 2H^2] \\ & - 2[(n-2)\operatorname{tr}(H)H + \operatorname{tr}(H)^2\operatorname{id}] \\ & + 2[(n-2)H^2 + |H|^2\operatorname{id}] \\ & = 2|H|^2\operatorname{id} - 2\operatorname{tr}(H)^2\operatorname{id}. \end{aligned}$$

This shows that the trace-free Ricci part of T vanishes.

Step 2: We claim that the condition $S \in \text{PIC2}$ is preserved. It suffices to show that the term

$$\begin{aligned} T &:= (n-2)H \otimes H - 2\operatorname{tr}(H)H \otimes \operatorname{id} + 2H^2 \otimes \operatorname{id} \\ &+ \frac{2}{\sigma(n-2\sigma)}\operatorname{tr}(H)^2\operatorname{id} \otimes \operatorname{id} - \frac{2-\sigma}{\sigma}|H|^2\operatorname{id} \otimes \operatorname{id} \end{aligned}$$

has nonnegative curvature operator. A straightforward calculation gives

$$\begin{aligned} T &= (n-2)A \otimes A + 2A^2 \otimes \operatorname{id} - 2\operatorname{tr}(A)A \otimes \operatorname{id} \\ &+ \frac{1}{\sigma^2}\operatorname{tr}(A)^2\operatorname{id} \otimes \operatorname{id} - \frac{2-\sigma}{\sigma}|A|^2\operatorname{id} \otimes \operatorname{id}, \end{aligned}$$

where $A := \frac{1}{n-2\sigma}\operatorname{tr}(H)\operatorname{id} - H \geq 0$. To show that T has nonnegative curvature operator, it therefore suffices to show that

$$\begin{aligned} & (n-2)a_i a_j + a_i^2 + a_j^2 - \operatorname{tr}(A)(a_i + a_j) \\ & + \frac{1}{\sigma^2}\operatorname{tr}(A)^2 - \frac{2-\sigma}{\sigma}|A|^2 \geq 0 \end{aligned}$$

for $i \neq j$, where a_i denotes the i -th eigenvalue of A .

We now verify this inequality. Since $A \geq 0$, we have

$$\operatorname{tr}(A)^2 - |A|^2 = \sum_{p \neq q} a_p a_q \geq 0$$

and

$$\begin{aligned} & (\operatorname{tr}(A) - a_i - a_j)^2 - (|A|^2 - a_i^2 - a_j^2) \\ &= \sum_{p, q \in \{1, \dots, n\} \setminus \{i, j\}, p \neq q} a_p a_q \geq 0 \end{aligned}$$

for $i \neq j$. At this point, we distinguish two cases:

Case 1: Suppose first that $\sigma \in (0, \frac{4}{3}]$. In this case, we have

$$\begin{aligned}
& (n-2) a_i a_j + a_i^2 + a_j^2 - \operatorname{tr}(A) (a_i + a_j) \\
& + \frac{1}{\sigma^2} \operatorname{tr}(A)^2 - \frac{2-\sigma}{\sigma} |A|^2 \\
& = (n-3) a_i a_j + \frac{(1-\sigma)^2}{\sigma^2} \operatorname{tr}(A)^2 \\
& + \frac{4-3\sigma}{2\sigma} (\operatorname{tr}(A)^2 - |A|^2) \\
& + \frac{1}{2} \left((\operatorname{tr}(A) - a_i - a_j)^2 - (|A|^2 - a_i^2 - a_j^2) \right) \\
& \geq 0,
\end{aligned}$$

as desired.

Case 2: Suppose now that $\sigma \in [\frac{4}{3}, 2]$. In this case, we obtain

$$\begin{aligned}
& (n-2) a_i a_j + a_i^2 + a_j^2 - \operatorname{tr}(A) (a_i + a_j) \\
& + \frac{1}{\sigma^2} \operatorname{tr}(A)^2 - \frac{2-\sigma}{\sigma} |A|^2 \\
& = \left(n-4 + 2 \frac{2-\sigma}{\sigma} \right) a_i a_j + \frac{(2-\sigma)^2}{4\sigma^2} \operatorname{tr}(A)^2 \\
& + \frac{3\sigma-4}{\sigma} \left(\frac{1}{2} \operatorname{tr}(A) - a_i - a_j \right)^2 \\
& + \frac{2-\sigma}{\sigma} \left((\operatorname{tr}(A) - a_i - a_j)^2 - (|A|^2 - a_i^2 - a_j^2) \right) \\
& \geq 0.
\end{aligned}$$

This proves the claim.

Step 3: In the next step, we show that the condition $\operatorname{tr}(H) \operatorname{id} - (n-2\sigma) H \geq 0$ is preserved. Using the fact that $\operatorname{Ric}_0(S) = 0$, we compute

$$\begin{aligned}
& \frac{d}{dt} (\operatorname{tr}(H) \operatorname{id} - (n-2\sigma) H) \\
& = 2 S * (\operatorname{tr}(H) \operatorname{id} - (n-2\sigma) H) + \frac{2}{n} \operatorname{scal}(S) (\operatorname{tr}(H) \operatorname{id} - (n-2\sigma) H) \\
& + 4 \operatorname{tr}(H) (\operatorname{tr}(H) \operatorname{id} - (n-2\sigma) H) + 4 (n-2\sigma) H^2 - \frac{4}{n-2\sigma} \operatorname{tr}(H)^2 \operatorname{id}.
\end{aligned}$$

This implies that the condition $\operatorname{tr}(H) \operatorname{id} - (n-2\sigma) H \geq 0$ is preserved.

Step 4: We next show that the condition $\operatorname{tr}(H) - \theta \operatorname{scal}(S) \geq 0$ is preserved if θ is sufficiently small. This is trivial if $\theta = 0$, so we will only consider the

case $\theta > 0$. We clearly have

$$\begin{aligned}
\frac{d}{dt} \text{scal}(S) &\leq 2 |\text{Ric}(S)|^2 + C(n) \text{tr}(H)^2 \\
&\quad + \frac{4(n-1)}{\sigma} \text{tr}(H)^2 - \frac{4n(n-1)}{\sigma} |H|^2 \\
&= \frac{2}{n} \text{scal}(S)^2 + C(n) \text{tr}(H)^2 \\
&\quad + \frac{4(n-1)}{\sigma} \text{tr}(H)^2 - \frac{4n(n-1)}{\sigma} |H|^2
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dt} \text{tr}(H) &\geq 2 \langle \text{Ric}(S), H \rangle + \frac{2}{n} \text{scal}(S) \text{tr}(H) - C(n) \text{tr}(H)^2 \\
&\quad - \frac{2}{\sigma} \text{tr}(H)^2 + \frac{2n}{\sigma} |H|^2 \\
&= \frac{4}{n} \text{scal}(S) \text{tr}(H) - C(n) \text{tr}(H)^2 \\
&\quad - \frac{2}{\sigma} \text{tr}(H)^2 + \frac{2n}{\sigma} |H|^2.
\end{aligned}$$

Hence, if $\text{tr}(H) - \theta \text{scal}(S) = 0$, then we obtain

$$\begin{aligned}
&\frac{d}{dt} (\text{tr}(H) - \theta \text{scal}(S)) \\
&\geq \frac{4}{n} \text{scal}(S) \text{tr}(H) - \frac{2}{n} \theta \text{scal}(S)^2 - C(n) (1 + \theta) \text{tr}(H)^2 \\
&\quad + \frac{2}{\sigma} (1 + 2(n-1)\theta) (n |H|^2 - \text{tr}(H)^2) \\
&\geq \frac{2}{n\theta} \text{tr}(H)^2 - C(n) (1 + \theta) \text{tr}(H)^2,
\end{aligned}$$

and the right hand side is strictly positive if θ is sufficiently small. Hence, if θ is small enough, then the condition $\text{tr}(H) - \theta \text{scal}(S) \geq 0$ is preserved.

Step 5: We now show that the sum $S + H \oslash \text{id}$ evolves by the Hamilton ODE. Using the fact that $\text{Ric}_0(S) = 0$, we obtain $\text{Ric}(S) \oslash H = \frac{1}{n} \text{scal}(S) H \oslash$

id. This gives

$$\begin{aligned}
& \frac{d}{dt}(S + H \otimes \text{id}) \\
&= Q(S) + (n-2)H \otimes H - 2\text{tr}(H)H \otimes \text{id} + 2H^2 \otimes \text{id} \\
&+ \frac{2}{\sigma(n-2\sigma)}\text{tr}(H)^2 \text{id} \otimes \text{id} - \frac{2-\sigma}{\sigma}|H|^2 \text{id} \otimes \text{id} \\
&+ 2(S * H) \otimes \text{id} + \frac{2}{n}\text{scal}(S)H \otimes \text{id} + 4\text{tr}(H)H \otimes \text{id} - 4H^2 \otimes \text{id} \\
&- \frac{2}{\sigma(n-2\sigma)}\text{tr}(H)^2 \text{id} \otimes \text{id} + \frac{2}{\sigma}|H|^2 \text{id} \otimes \text{id} \\
&= Q(S) + 2\text{Ric}(S) \otimes H + 2(S * H) \otimes \text{id} \\
&+ (n-2)H \otimes H + 2\text{tr}(H)H \otimes \text{id} \\
&- 2H^2 \otimes \text{id} + |H|^2 \text{id} \otimes \text{id} \\
&= Q(S + H \otimes \text{id})
\end{aligned}$$

in view of Lemma 2.1. This shows that the cone $\mathcal{C}_{\sigma,\theta}$ is invariant under the Hamilton ODE for each $\sigma \in (0, 2]$ and $\theta \in [0, \bar{\theta}]$.

Step 6: Finally, to prove the transversality statement, we assume that $\sigma \in (0, 1) \cup (1, 2)$ and $\theta \in (0, \bar{\theta})$. If $R = S + H \otimes \text{id} \in \mathcal{C}_{\sigma,\theta} \setminus \{0\}$, then $\text{tr}(H) > 0$, hence $\text{tr}(A) > 0$. The argument in Step 3 now implies that the term T has strictly positive curvature operator. Consequently, the cone $\mathcal{C}_{\sigma,\theta}$ is transversally invariant away from 0. This completes the proof.

As a corollary of Theorem 2.5, we obtain a higher-dimensional version of the Hamilton-Ivey pinching estimate for three-dimensional Ricci flow.

Theorem 2.6. *Let us fix real numbers $\sigma_0 \in (1, 2)$ and $\theta \in (0, \bar{\theta})$. Then there exists a concave and increasing function $f : [0, \infty) \rightarrow [0, \infty)$ such that $f(s) = \frac{s}{n-2\sigma_0}$ for s sufficiently small, $\lim_{s \rightarrow \infty} \frac{f(s)}{s} = \frac{1}{n-2}$, and the set*

$$\begin{aligned}
& \{R = S + H \otimes \text{id} : S \in \text{PIC2}, \text{Ric}_0(S) = 0, \\
& f(\text{tr}(H)) \text{id} - H \geq 0, \\
& \text{tr}(H) - \theta \text{scal}(S) \geq 0\}
\end{aligned}$$

is preserved by the Hamilton ODE.

Proof. By Theorem 2.5, the cones $\mathcal{C}_{\sigma,\theta}$ are transversally invariant from 0 for all $\sigma \in (1, 2)$ and all $\theta \in (0, \bar{\theta})$. We now consider a set of the form

$$F := \mathcal{C}_{\sigma,\theta} \cap \bigcap_{j \in \mathbb{N}} \{R : R + 2^{j-1} \text{id} \otimes \text{id} \in \mathcal{C}_{\sigma_j,\theta}\},$$

where σ_j is a decreasing sequence of real numbers such that $1 < \sigma_j < \sigma_0$. Arguing as in Theorem 4.1 in [1] or Proposition 17 in [6], we can choose the

sequence σ_j such that the set F is invariant under the Hamilton ODE and $\lim_{j \rightarrow \infty} \sigma_j = 1$. On the other hand, we may write

$$\begin{aligned} F = \{ R = S + H \oslash \text{id} : S \in \text{PIC2}, \text{Ric}_0(S) = 0, \\ f(\text{tr}(H)) \text{id} - H \geq 0, \\ \text{tr}(H) - \theta \text{scal}(S) \geq 0 \}, \end{aligned}$$

where $f(s) := \min \left\{ \frac{s}{n-2\sigma_0}, \inf_{j \in \mathbb{N}} \frac{s+2^j \sigma_j}{n-2\sigma_j} \right\}$. Clearly, f is concave, $f(s) = \frac{s}{n-2\sigma_0}$ for s sufficiently small, and $\lim_{s \rightarrow \infty} \frac{f(s)}{s} = \frac{1}{n-2}$. This proves the assertion.

Corollary 2.7. *Let (M, g_0) be a compact n -dimensional manifold whose curvature tensor lies in the cone $\mathcal{C}_{\sigma, \theta}$ at each point. Let $g(t)$ denote the solution of the Ricci flow with initial metric g_0 . Then the curvature tensor of any blow-up limit lies in the cone $\mathcal{C}_{1, \theta}$.*

3. SPLITTING THEOREMS

In this section, we collect two splitting theorems, which will play a key role in the subsequent arguments. The following result is a higher-dimensional analogue of Hamilton's crucial splitting theorem (cf. [17], Theorem C5.1):

Proposition 3.1. *Let $(M, g(t))$, $t \in (0, T)$, be a (possibly incomplete) solution to the Ricci flow whose curvature tensor lies in the cone $\mathcal{C}_{1, \theta}$. Moreover, suppose that there exists a point (p_0, t_0) in space-time with the property that the curvature tensor lies on the boundary of the cone $\mathcal{C}_{1, \theta}$. Then the manifold $(M, g(t_0))$ is either flat or it is locally isometric to a subset of the round cylinder $S^{n-1} \times \mathbb{R}$.*

Proof. By assumption, the curvature tensor R lies in the cone $\mathcal{C}_{1, \theta}$. This is equivalent to saying that

$$R - \frac{1}{n-2} \text{Ric}_0 \oslash \text{id} - \frac{1}{n} \frac{\theta}{1 + 2(n-1)\theta} \text{scal id} \oslash \text{id} \in \text{PIC2}$$

and

$$|v|^2 R - \frac{1}{n-2} |v|^2 \text{Ric}_0 \oslash \text{id} - \frac{1}{2} \text{Ric}_0(v, v) \text{id} \oslash \text{id} \in \text{PIC2}$$

for every tangent vector v . Let E denote the total space of the vector bundle $TM \oplus T^{\mathbb{C}}M \oplus T^{\mathbb{C}}M$ over $M \times (0, T)$. Moreover, let Ω be the set of all triplets $(v, z, w) \in E$ such that $v \neq 0$ and z, w are linearly independent. We define a function $\varphi : E \rightarrow \mathbb{R}$ by

$$\begin{aligned} \varphi(v, z, w) := |v|^2 R(z, w, \bar{z}, \bar{w}) - \frac{1}{n-2} |v|^2 (\text{Ric}_0 \oslash \text{id})(z, w, \bar{z}, \bar{w}) \\ - \frac{1}{2} \text{Ric}_0(v, v) (\text{id} \oslash \text{id})(z, w, \bar{z}, \bar{w}). \end{aligned}$$

Clearly, φ is nonnegative since $R \in \mathcal{C}_{1,\theta}$. Moreover, φ satisfies

$$D_t \varphi \geq \mathcal{L} \varphi + L \inf_{|\xi| \leq 1} (D^2 \varphi)(\xi, \xi) - L |d\varphi| - L \varphi,$$

on Ω , where \mathcal{L} denotes the horizontal Laplacian on E and L is a positive function in $L_{loc}^\infty(\Omega)$. Using Bony's strict maximum principle for degenerate elliptic equations, we conclude that the set $\{(v, z, w) \in \Omega : \varphi(v, z, w) = 0\}$ is invariant under parallel transport along spatial curves.

We now consider the given time t_0 , and define

$$S := \{v \in TM : \text{there exist linearly independent} \\ \text{vectors } z, w \in T^\mathbb{C}M \text{ such that } \varphi(v, z, w) = 0\}.$$

We have shown that this set is invariant under parallel transport. Moreover, it is easy to see that each fiber of S is a linear subspace of the tangent space of $(M, g(t_0))$. Indeed, the fiber of S over any given point either equals $\{0\}$, or it equals the eigenspace corresponding to the largest eigenvalue of the Ricci tensor at that point. Consequently, S defines a parallel subbundle of TM . There are three possibilities:

Case 1: The parallel subbundle S has rank n . In this case, we have $\text{Ric}_0 = 0$ and $R - \frac{1}{n-2} \text{Ric}_0 \otimes \text{id} \in \partial \text{PIC2}$ at each point on $(M, g(t_0))$. Since $R - \frac{1}{n-2} \text{Ric}_0 \otimes \text{id} - \frac{1}{n} \frac{\theta}{1+2(n-1)\theta} \text{scal id} \otimes \text{id} \in \text{PIC2}$, we conclude that the scalar curvature of $(M, g(t_0))$ vanishes. Thus, $(M, g(t_0))$ is flat.

Case 2: The parallel subbundle S has rank $n-1$. In this case, $(M, g(t_0))$ locally splits as a product of two manifolds of dimension $n-1$ and 1. By Proposition 2.3, $(M, g(t_0))$ is locally isometric to a subset of the round cylinder $S^{n-1} \times \mathbb{R}$.

Case 3: The parallel subbundle S has rank $k \in \{1, \dots, n-2\}$. In this case, $(M, g(t_0))$ locally splits as a product of two manifolds of dimension k and $n-k$. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of the tangent space at (p_0, t_0) with the property that the fiber of S is spanned by $\{e_1, \dots, e_k\}$. Clearly, $R(e_1, e_n, e_1, e_n) = 0$ since $(M, g(t_0))$ locally splits as a product. This implies

$$\begin{aligned} 0 &\leq \varphi(e_1, e_n; e_1) \\ &= R(e_1, e_n, e_1, e_n) - \frac{1}{n-2} (\text{Ric}_0(e_1, e_1) + \text{Ric}_0(e_n, e_n)) - \text{Ric}_0(e_1, e_1) \\ &= -\frac{1}{n-2} ((n-1) \text{Ric}_0(e_1, e_1) + \text{Ric}_0(e_n, e_n)). \end{aligned}$$

On the other hand, we know that the eigenspace corresponding to the largest eigenvalue of the Ricci tensor is spanned by $\{e_1, \dots, e_k\}$. Since $k \leq n-2$, we obtain

$$(n-1) \text{Ric}_0(e_1, e_1) + \text{Ric}_0(e_n, e_n) > \sum_{i=1}^{n-1} \text{Ric}_0(e_i, e_i) + \text{Ric}_0(e_n, e_n) = 0.$$

This is a contradiction.

Case 4: The parallel subbundle S has rank 0. In this case,

$$|v|^2 R - \frac{1}{n-2} |v|^2 \text{Ric}_0 \otimes \text{id} - \frac{1}{2} \text{Ric}_0(v, v) \text{id} \otimes \text{id}$$

lies in the interior of the PIC2 cone for each $v \neq 0$. Moreover, it is easy to see that

$$R - \frac{1}{n-2} \text{Ric}_0 \otimes \text{id} - \frac{1}{n} \frac{\theta}{1 + 2(n-1)\theta} \text{scal} \text{id} \otimes \text{id}$$

lies in the interior of the cone PIC2 for all $t \in (0, T)$. Putting these facts together, we conclude that the curvature tensor of $(M, g(t_0))$ lies in the interior of the cone $\mathcal{C}_{1,\theta}$ at each point. This contradicts the assumption that the curvature tensor lies on the boundary of $\mathcal{C}_{1,\theta}$ at the point (p_0, t_0) . This completes the proof.

The next result is an adaptation of a result of Perelman [26].

Proposition 3.2. *Let (M, g) be a complete noncompact manifold whose curvature tensor lies in the interior of the cone $\mathcal{C}_{1,\theta}$. Let us fix a point $p \in M$ and let p_j be a sequence of points such that $d(p, p_j) \rightarrow \infty$. Moreover, let λ_j be a sequence of positive real number such that $\lambda_j d(p, p_j)^2 \rightarrow \infty$. If the rescaled manifolds $(M, \lambda_j g, p_j)$ converge in the Cheeger-Gromov sense to a smooth, non-flat limit Y , then Y is isometric to $S^{n-1} \times \mathbb{R}$.*

Proof. We follow the exposition in [9], Lemma 3.1. Our assumptions imply that (M, g) has positive sectional curvature. Hence, by a result of Perelman, the limit Y splits off a line (see also [9], Proposition 2.3). Consequently, we have $Y = X \times \mathbb{R}$ for some $(n-1)$ -dimensional manifold X . Since the curvature of Y lies in the cone $\mathcal{C}_{1,\theta}$, we conclude that X has constant curvature by Proposition 2.3. Consequently, X is isometric to a space form S^{n-1}/Γ . An argument due to Hamilton (see Theorem A.1) then implies that the cross-section X is incompressible in M . Since M is diffeomorphic to \mathbb{R}^n by the soul theorem (cf. [8]), this implies that Γ is trivial. Thus X is isometric to S^{n-1} , and Y is isometric to $S^{n-1} \times \mathbb{R}$, as claimed.

For later use, we also recall the following result due to Perelman:

Proposition 3.3 (cf. G. Perelman [26]). *Let (M, g) be a complete Riemannian manifold whose curvature tensor lies in the interior of the cone $\mathcal{C}_{1,\theta}$. Moreover, suppose that (M, g) is κ -noncollapsed, and that the covariant derivatives of the Riemann curvature tensor satisfy the pointwise estimates $|DR| \leq \eta \text{scal}^{\frac{3}{2}}$ and $|D^2 R| \leq \eta \text{scal}^2$ at all points where the scalar curvature is sufficiently large. Then (M, g) has bounded curvature.*

Proof. Suppose that (M, g) does not have bounded curvature. Using a standard point-picking lemma, we can find a sequence of points x_j such that

$Q_j := \text{scal}(x_j) \geq j$ and

$$\sup_{x \in B(x_j, j Q_j^{-\frac{1}{2}})} \text{scal}(x) \leq 4Q_j.$$

We now dilate the manifold (M, g) around the point x_j by the factor Q_j . Using the noncollapsing assumption and the curvature derivative estimates, we are able to take a limit in $C_{loc}^{3,\alpha}$. The limit manifold (M^∞, g^∞) is of class $C^{3,\alpha}$, is complete and has bounded curvature. By a result of Perelman, the limit Y splits off a line (see also [9], Proposition 2.3). As above, we conclude that the limit must be a round cylinder. Consequently, (M, g) contains a sequence of necks with radii converging to 0. But this is impossible in a manifold with positive sectional curvature (see [9], Proposition 2.2, for a detailed proof).

4. ANCIENT κ -SOLUTIONS WITH θ -PINCHED CURVATURE

In this section, we sketch how the arguments in Section 11 of Perelman's first paper extend to higher dimensions. The adaptation to the four-dimensional case was done in [9]. Through this section, we fix an integer $n \geq 5$ and an arbitrary positive constant $\theta \in (0, \bar{\theta})$. Our goal is to analyze ancient solutions whose curvature tensor lies in the cone $\mathcal{C}_{1,\theta}$. We will use the following terminology:

Definition 4.1. An ancient κ -solution with θ -pinched curvature is a non-flat ancient solution of Ricci flow of dimension n which is complete with bounded curvature; is κ -noncollapsed on all scales; and has curvature in the cone $\mathcal{C}_{1,\theta}$.

By [4], Hamilton's trace Harnack inequality holds in this setting:

Theorem 4.2 (R. Hamilton [15]. Corollary 1.2; S. Brendle [4], Corollary 2). *Let $(M, g(t))$ be an ancient κ -solution with θ -pinched curvature. Then*

$$\frac{\partial}{\partial t} \text{scal} + 2 \langle \nabla \text{scal}, v \rangle + 2 \text{Ric}(v, v) \geq 0$$

for every tangent vector v . In particular, the scalar curvature is monotone increasing at each point.

Proof. This was established in a seminal paper of Hamilton [15] under the stronger assumption of nonnegative curvature operator. In [4], we showed that Hamilton's Harnack estimate holds under the weaker assumption that the curvature tensor lies in the cone PIC2. Since the cone $\mathcal{C}_{1,\theta}$ is contained in the cone PIC2, the assertion follows.

Integrating the trace Harnack inequality along paths in space-time gives the following result:

Corollary 4.3 (cf. [15], [4]). *Let $(M, g(t))$ be an ancient κ -solution with θ -pinched curvature. Then*

$$\text{scal}(x_1, t_1) \leq \exp\left(\frac{d_{g(t_1)}(x_1, x_2)^2}{2(t_2 - t_1)}\right) \text{scal}(x_2, t_2)$$

whenever $t_1 < t_2$.

We next recall a key result from Perelman's first paper:

Theorem 4.4 (cf. G. Perelman [26], Corollary 11.6). *For every $w > 0$, there exist positive constants B and C with the following property: Suppose that $(M, g(t))$, $t \in [0, T]$, is a solution to the Ricci flow so that the ball $B_{g(T)}(x_0, r_0)$ is compactly contained in M . Moreover, suppose that for each $t \in [0, T]$, the curvature tensor of $g(t)$ lies in the PIC2 cone, and $\text{vol}_{g(t)}(B_{g(t)}(x_0, r_0)) \geq wr_0^n$. Then $\text{scal}(x, t) \leq Cr_0^{-2} + Bt^{-1}$ for all $t \in (0, T]$ and all $x \in B_{g(t)}(x_0, \frac{1}{4}r_0)$.*

Proof. The only difference to the statement in Perelman's paper is that we have replaced the assumption that $g(t)$ has nonnegative curvature operator by the weaker PIC2 assumption. This does not affect the proof.

In the next step, we establish an analogue of Perelman's longrange curvature estimate for ancient κ -solutions in dimension 3. Perelman's estimate was adapted to the four-dimensional case in [9], Proposition 3.3.

Theorem 4.5 (cf. G. Perelman [26], Section 11.7). *Given $\kappa > 0$, there exists a large positive constant η and a positive function $\omega : [0, \infty) \rightarrow (0, \infty)$ (depending on κ) such that the following holds: Let $(M, g(t))$ be an ancient κ -solution with θ -pinched curvature. Then*

$$\text{scal}(x, t) \leq \text{scal}(y, t) \omega(\text{scal}(y, t) d_{g(t)}(x, y)^2)$$

for all points $x, y \in M$ and all t . Furthermore, we have the pointwise estimates $|DR| \leq \eta \text{scal}^{\frac{3}{2}}$ and $|D^2R| \leq \eta \text{scal}^2$, where R denotes the Riemann curvature tensor.

Proof. We sketch the argument, following Section 11.7 in Perelman's paper [26] (see also [9], Proposition 3.3). The second statement follows immediately from the first statement together with Shi's interior derivative estimate. Thus, it suffices to prove the first statement. Without loss of generality, we may assume that $t = 0$. Moreover, by scaling we may assume that $\text{scal}(y, 0) = 1$. Let A denote the set of all points $x \in M$ such that $\text{scal}(x, 0) + 1 \geq d_{g(0)}(y, x)^{-2}$. Moreover, let $z \in A$ be a point which has minimal distance from y with respect to the metric $g(0)$ among all points in A . Clearly, $\text{scal}(z, 0) + 1 = d_{g(0)}(y, z)^{-2}$.

Let p denote the mid-point of the minimizing geodesic in $(M, g(0))$ joining y and z . It follows from the definition of z that $B_{g(0)}(p, \frac{1}{4} d_{g(0)}(y, z)) \cap A = \emptyset$.

In other words, $\text{scal}(x, 0) + 1 \leq d_{g(0)}(y, x)^{-2}$ for all $x \in B_{g(0)}(p, \frac{1}{4} d_{g(0)}(y, z))$. Using the Harnack inequality, we obtain

$$\begin{aligned} \sup_{x \in B_{g(t)}(p, \frac{1}{4} d_{g(0)}(y, z))} \text{scal}(x, t) &\leq \sup_{x \in B_{g(0)}(p, \frac{1}{4} d_{g(0)}(y, z))} \text{scal}(x, 0) \\ &\leq 16 d_{g(0)}(y, z)^{-2} \end{aligned}$$

for all $t \in (-\infty, 0]$. The noncollapsing property gives

$$\text{vol}_{g(t)}(B_{g(t)}(p, \frac{1}{4} d_{g(0)}(y, z))) \geq \kappa \left(\frac{1}{4} d_{g(0)}(y, z)\right)^n$$

for all $t \in (-\infty, 0]$. Using Theorem 4.4, we conclude that

$$\sup_{x \in B_{g(0)}(p, r)} \text{scal}(x, 0) \leq d_{g(0)}(y, z)^{-2} \omega(d_{g(0)}(y, z)^{-1} r)$$

for some positive and increasing function ω . The Harnack inequality gives

$$\begin{aligned} \sup_{x \in B_{g(0)}(p, d_{g(0)}(y, z))} \text{scal}(x, t) &\leq \sup_{x \in B_{g(0)}(p, d_{g(0)}(y, z))} \text{scal}(x, 0) \\ &\leq d_{g(0)}(y, z)^{-2} \omega(1) \end{aligned}$$

for all $t \in (-\infty, 0]$. In particular, there exists a positive constant β such that $d_{g(t)}(y, z) \leq 2 d_{g(0)}(y, z)$ for all $t \in [-\beta d_{g(0)}(y, z)^2, 0]$. Moreover, by choosing β sufficiently small, we can arrange that $\text{scal}(z, 0) \leq \text{scal}(z, t) + \frac{1}{2} d_{g(0)}(y, z)^{-2}$ for all $t \in [-\beta d_{g(0)}(y, z)^2, 0]$. (This follows from Shi's interior derivative estimate.) Applying the Harnack inequality with $t = -\beta d_{g(0)}(y, z)^2$ yields

$$\begin{aligned} \frac{1}{2} d_{g(0)}(y, z)^{-2} - 1 &= \text{scal}(z, 0) - \frac{1}{2} d_{g(0)}(y, z)^{-2} \\ &\leq \text{scal}(z, t) \\ &\leq \exp\left(-\frac{d_{g(t)}(y, z)^2}{2t}\right) \text{scal}(y, 0) \\ &\leq \exp\left(-\frac{2 d_{g(0)}(y, z)^2}{t}\right) \text{scal}(y, 0) \\ &= \exp\left(\frac{2}{\beta}\right). \end{aligned}$$

This implies $d_{g(0)}(y, z)^{-2} \leq 4 e^{\frac{2}{\beta}}$. Moreover, we have $d_{g(0)}(y, p) = \frac{1}{2} d_{g(0)}(y, z) \leq \frac{1}{2}$. Putting these facts together, we obtain

$$\begin{aligned} \sup_{x \in B_{g(0)}(y, r)} \text{scal}(x, 0) &\leq \sup_{x \in B_{g(0)}(p, r+1)} \text{scal}(x, 0) \\ &\leq d_{g(0)}(y, z)^{-2} \omega(d_{g(0)}(y, z)^{-1} (r+1)) \\ &\leq 4 e^{\frac{2}{\beta}} \omega(2 e^{\frac{1}{\beta}} (r+1)). \end{aligned}$$

This proves the assertion.

As a corollary, we obtain a higher dimensional version of Perelman's compactness theorem for ancient κ -solutions:

Corollary 4.6 (cf. G. Perelman [26], Section 11.7). *Let $(M^{(j)}, g^{(j)}(t))$ be a sequence of ancient κ -solution with θ -pinched curvature. Suppose that each solution is defined for $t \in (-\infty, 0]$, and that $\text{scal}(x_j, 0) = 1$ for some point x_j . Then, after passing to a subsequence if necessary, the sequence $(M^{(j)}, g^{(j)}(t), x_j)$ converges in the Cheeger-Gromov sense to an ancient κ -solution with θ -pinched curvature.*

Proof. It follows from Theorem 4.5 that, after passing to a subsequence if necessary, the sequence $(M^{(j)}, g^{(j)}(t), x_j)$ converges in the Cheeger-Gromov sense to a smooth ancient solution $(M^\infty, g^\infty(t))$. Clearly, $(M^\infty, g^\infty(t))$ is κ -noncollapsed and has θ -pinched curvature. Moreover, Theorem 4.5 implies that $(M^{(j)}, g^{(j)}(t))$ satisfies $|DR| \leq \eta \text{scal}^{\frac{3}{2}}$ and $|D^2R| \leq \eta \text{scal}^2$, where η is a positive constant which does not depend on j . Hence, these estimates also hold on the limiting ancient solution $(M^\infty, g^\infty(t))$. Proposition 3.3 now implies that the limiting ancient solution has bounded curvature.

Theorem 4.7 (cf. G. Perelman [26], Corollary 11.8; Chen-Zhu [9], Proposition 3.4). *Given $\varepsilon > 0$, there exists a positive constant $C = C(n, \theta, \varepsilon)$ such that the following holds: Let $(M, g(t))$ be a non-compact, non-flat ancient κ -solution with θ -pinched curvature which is not locally isometric to a round cylinder. Given any point (x_0, t_0) in space-time there exists an open neighborhood B of x_0 such that $B_{g(t_0)}(x_0, C_1^{-1} \text{scal}(x_0, t_0)^{-\frac{1}{2}}) \subset B \subset B_{g(t_0)}(x_0, C_1 \text{scal}(x_0, t_0)^{-\frac{1}{2}})$ and $C_2^{-1} \text{scal}(x_0, t_0) \leq \text{scal}(x, t_0) \leq C_2 \text{scal}(x_0, t_0)$ for all $x \in B$. Moreover, B falls into one of the following three categories:*

- *B is an ε -neck.*
- *B is an ε -cap in the sense that B is diffeomorphic to a ball and the boundary ∂B is a cross-sectional sphere of an ε -neck.*

In particular, $(M, g(t_0))$ is κ_0 -noncollapsed for some universal constant $\kappa_0 = \kappa_0(n)$.

Proof. Without loss of generality, we may assume that $t_0 = 0$. Moreover, we may assume that x_0 does not lie at the center of an ε -neck. Since $(M, g(t))$ is not locally isometric to a round cylinder, it follows from Proposition 3.1 that the curvature tensor of $(M, g(t))$ lies in the interior of the cone $\mathcal{C}_{1, \theta}$. In particular, M is diffeomorphic to S^n by the soul theorem. Let M_ε denote the set of all points in $(M, g(0))$ which do not lie at the center of an $\frac{\varepsilon}{2}$ -neck. Clearly, $x_0 \in M_\varepsilon$ since x_0 does not lie at the center of an ε -neck.

Step 1: We first show that M_ε is compact. Suppose this is false. Then there exists a sequence of points $x_j \in M_\varepsilon$ such that $d(x_0, x_j) \rightarrow \infty$. Since $\text{scal}(x_0, 0) > 0$, Theorem 4.5 implies that $\liminf_{j \rightarrow \infty} \lambda_j d_{g(0)}(x_0, x_j)^2 = \infty$, where $\lambda_j = \text{scal}(x_j, 0)$. Using Corollary 4.6, we conclude that the rescaled manifolds $(M, \lambda_j g(0), x_j)$ converge in the Cheeger-Gromov sense

to a smooth non-flat limit. Since $\lambda_j d_{g(0)}(x_0, x_j)^2 \rightarrow \infty$, Proposition 3.2 implies that the limit is isometric to a cylinder. In particular, x_j lies on an $\frac{\varepsilon}{2}$ -neck if j is sufficiently large. This contradicts the fact that $x_j \in M_\varepsilon$.

Thus, M_ε is compact. In particular, $M_\varepsilon \neq M$. Since $M_\varepsilon \neq \emptyset$, it follows that $\partial M_\varepsilon \neq \emptyset$.

Step 2: Let us consider an arbitrary point $y \in \partial M_\varepsilon$. Clearly, y lies on an ε -neck in $(M, g(0))$. Hence, there exists a universal constant $\beta = \beta(n) > 0$ such that

$$\text{vol}_{g(t)}(B_{g(t)}(y, \text{scal}(y, 0)^{-\frac{1}{2}})) \geq \beta \text{scal}(y, 0)^{-\frac{n}{2}}$$

for all $t \in [-\beta \text{scal}(y, 0)^{-1}, 0]$. Using Theorem 4.4, we conclude that

$$\text{scal}(x, 0) \leq \text{scal}(y, 0) \omega(\text{scal}(y, 0) d_{g(0)}(x, y)^2)$$

for all $x \in M$, where ω is a positive function that does not depend on κ .

Step 3: We again consider an arbitrary point $y \in \partial M_\varepsilon$. Recall that y lies on an ε -neck in $(M, g(0))$. Let Σ_y denote the leaf in Hamilton's CMC foliation which passes through the point y . Since M is diffeomorphic to \mathbb{R}^n , there is a unique bounded connected component of $M \setminus \Sigma_y$, and this connected component is diffeomorphic to a ball in view of the solution of the Schoenflies conjecture in dimension $n \neq 4$. Let us denote this connected component by Ω_y .

We claim that $\text{scal}(y, 0) \text{diam}_{g(0)}(\Omega_y)^2 \leq C$, where C depends on ε , but not on κ . To prove this, we argue by contradiction. Suppose that $(M^{(j)}, g^{(j)}(t))$ is a sequence of non-compact, non-flat ancient κ -solutions with θ -pinched curvature which are not locally isometric to a round cylinder. Moreover, suppose that y_j is a sequence of points such that $y_j \in \partial M_\varepsilon^{(j)}$ and $\text{scal}(y_j, 0) \text{diam}_{g^{(j)}(0)}(\Omega_{y_j})^2 \rightarrow \infty$, where Ω_{y_j} denotes the region in $(M^{(j)}, g^{(j)}(0))$ which is bounded by the CMC sphere passing through y_j . We dilate the manifold $(M^{(j)}, g^{(j)}(0))$ around the point y_j by the factor $\text{scal}(y_j, 0)$. The curvature estimate established in Step 2 implies that, after passing to a subsequence if necessary, the rescaled manifolds converge to a smooth limit. Moreover, since $\text{scal}(y_j, 0) \text{diam}_{g^{(j)}(0)}(\Omega_{y_j})^2 \rightarrow \infty$, the limiting manifold has at least two ends. Consequently, the limiting manifold splits off a line by the Cheeger-Gromoll splitting theorem. Since the curvature tensor lies in the cone $\mathcal{C}_{1,\theta}$ at each point, the limit must be isometric to $(S^{n-1}/\Gamma) \times \mathbb{R}$. As above, Theorem A.1 implies that the cross section S^{n-1}/Γ is incompressible in M . Since M is diffeomorphic to \mathbb{R}^n , it follows that Γ is trivial. Thus, the limit is isometric to the round cylinder $S^{n-1} \times \mathbb{R}$. Hence, if j is sufficiently large, then y_j lies in the interior of the set $M_\varepsilon^{(j)}$. This contradicts our choice of y_j .

Step 4: Combining the curvature estimate in Step 2 with the diameter estimate in Step 3 gives $\text{scal}(x, 0) \leq C \text{scal}(y, 0)$ for all $y \in \partial M_\varepsilon$ and all $x \in \Omega_y$. Here, C is a positive constant that depends only on ε , but not on κ . Moreover, the Harnack inequality (cf. Corollary 4.3 above) implies that $\text{scal}(x, 0) \geq \frac{1}{C} \text{scal}(y, 0)$ for all $y \in \partial M_\varepsilon$ and all $x \in \Omega_y$, where C depends only on ε , but not on κ .

Step 5: Finally, we observe that the sets Ω_y are nested. More precisely, given two points $y, y' \in \partial M_\varepsilon$, we either have $\Omega_y \subset \Omega_{y'}$ or $\Omega_{y'} \subset \Omega_y$. Since ∂M_ε is compact, we can find a point $y_0 \in \partial M_\varepsilon$ such that $\Omega_y \subset \Omega_{y_0}$ for all $y \in \partial M_\varepsilon$. In particular, the set ∂M_ε is contained in the closure of Ω_{y_0} . Since M_ε is compact, it follows that M_ε is contained in the closure of Ω_{y_0} . In particular, we have $x_0 \in \Omega_{y_0}$. Since x_0 does not lie at the center of an ε -neck, the distance of x_0 to the boundary $\partial\Omega_{y_0} = \Sigma_{y_0}$ is bounded from below by $C^{-1} \text{scal}(x_0, 0)^{-\frac{1}{2}}$. Therefore, $B_{g(0)}(x_0, C^{-1} \text{scal}(x_0, 0)^{-\frac{1}{2}}) \subset \Omega_{y_0}$. On the other hand, the diameter bound in Step 3 gives $\Omega_{y_0} \subset B_{g(0)}(x_0, C \text{scal}(x_0, 0)^{-\frac{1}{2}})$. Thus, the set $B := \Omega_{y_0}$ has all the required properties.

In the next step, we state a universal noncollapsing property for ancient κ -solutions. This was first established by Perelman [27] in the three-dimensional case, and adapted to dimension 4 in [9].

Theorem 4.8 (cf. G. Perelman [26]; Chen-Zhu [9]). *There exists a constant $\kappa_0 = \kappa_0(n, \theta)$ such that the following holds: Let $(M, g(t))$ be an ancient κ -solution with θ -pinched curvature for some $\kappa > 0$. Then either $(M, g(t))$ is κ_0 -noncollapsed for all t ; or $(M, g(t))$ is a quotient of the round sphere S^n ; or $(M, g(t))$ is a noncompact quotient of the round cylinder $S^{n-1} \times \mathbb{R}$.*

Proof. We follow the arguments in [9]. If M is noncompact, the assertion is a consequence of Theorem 4.7. Hence, it remains to consider the case that M is compact. Let $(\bar{M}, \bar{g}(t))$ denote the asymptotic shrinking soliton of $(M, g(t))$ (cf. [26]).

Case 1: Suppose that \bar{M} is compact. It follows from Proposition 3.1 that the curvature tensor of \bar{M} lies in the interior of the cone $\mathcal{C}_{1, \theta}$. (A compact shrinking soliton cannot locally split as a product.) Consequently, \bar{M} is isometric to a metric quotient of the round sphere S^n . This directly implies that the flow $(M, g(t))$ is a metric quotient of S^n .

Case 2: Suppose next that \bar{M} is noncompact. We will show that \bar{M} is noncollapsed for some universal constant. By Theorem 4.7, the asymptotic shrinking soliton \bar{M} is either κ_0 -noncollapsed for some uniform κ_0 , or it is isometric to a metric quotient of the round cylinder. Suppose first that n is odd and $\bar{M} = (S^{n-1} \times \mathbb{R})/\Gamma$. In this case, there are only finitely many possibilities for the group Γ , and the resulting quotients are all noncollapsed with a universal constant. We next assume that n is even and $\bar{M} = (S^{n-1} \times \mathbb{R})/\Gamma$. By a result of Hamilton (cf. Theorem A.1), the center slice $(S^{n-1} \times \{0\})/\Gamma$ is incompressible in \bar{M} . Moreover, since n is even, the fundamental group of \bar{M} has order at most 2 by Synge's theorem. Again, this leaves only finitely many possibilities for the group Γ , and the resulting quotients $(S^{n-1} \times \mathbb{R})/\Gamma$ are noncollapsed with a universal constant. To summarize, we have shown that \bar{M} is noncollapsed with a universal constant. From this, we can deduce that $(M, g(t))$ is noncollapsed with some universal constant (see [9], Section 3.3 for details).

Using Theorem 4.7, we can draw the following conclusion:

Corollary 4.9 (cf. G. Perelman [27], Section 1.5). *Given $\varepsilon > 0$, we can find constants $C_1 = C_1(n, \theta, \varepsilon)$ and $C_2 = C_2(n, \theta, \varepsilon)$ with the following property: Suppose that $(M, g(t))$ is a non-flat ancient κ -solution with θ -pinched curvature. Then, for each point (x_0, t_0) in space-time there exists a neighborhood B of x_0 such that $B_{g(t_0)}(x_0, C_1^{-1} \text{scal}(x_0, t_0)^{-\frac{1}{2}}) \subset B \subset B_{g(t_0)}(x_0, C_1 \text{scal}(x_0, t_0)^{-\frac{1}{2}})$ and $C_2^{-1} \text{scal}(x_0, t_0) \leq \text{scal}(x, t_0) \leq C_2 \text{scal}(x_0, t_0)$ for all $x \in B$. Finally, B falls into one of the following three categories:*

- B is an ε -neck.
- B is an ε -cap in the sense that B is diffeomorphic to a ball and the boundary ∂B is a cross-sectional sphere of an ε -neck.
- B is a closed manifold diffeomorphic to S^n/Γ .
- B is an ε -quotient neck of the form $(S^{n-1} \times [-L, L])/\Gamma$.

Proof. We follow the arguments of Chen-Zhu [9] in the four-dimensional case. Without loss of generality, we may assume that $t_0 = 0$. By Theorem 4.8, $(M, g(t))$ is either κ_0 -noncollapsed for all t ; or $(M, g(t))$ is a quotient of the round sphere S^n ; or $(M, g(t))$ is a noncompact quotient of the round cylinder $S^{n-1} \times \mathbb{R}$. In the last two cases, the conclusion of Corollary 4.9 is trivially true. Hence, it suffices to prove the assertion under the assumption that $(M, g(t))$ is κ_0 -noncollapsed for all t . To that end, we argue by contradiction. Suppose that there exists a sequence of non-flat ancient κ_0 -solutions $(M^{(j)}, g^{(j)}(t))$ with θ -pinched curvature and a sequence of points $x_j \in M^{(j)}$ with the following properties:

- $(M^{(j)}, g^{(j)}(t))$ is not locally isometric to a round cylinder.
- If B is a neighborhood of x_j such that $B_{g(0)}(x_j, j^{-1} \text{scal}(x_j, 0)^{-\frac{1}{2}}) \subset B \subset B_{g(0)}(x_j, j \text{scal}(x_j, 0)^{-\frac{1}{2}})$ and $j^{-1} \text{scal}(x_j, 0) \leq \text{scal}(x, 0) \leq j \text{scal}(x_j, 0)$ for all $x \in B$, then B cannot fall into any of the three categories described above.

By scaling, we may assume that $\text{scal}(x_j, 0) = 1$ for each j . We now apply Corollary 4.6 to this sequence. Hence, after passing to a subsequence, we may assume that the sequence $(M^{(j)}, g^{(j)}(t), x_j)$ converges to a non-flat ancient κ_0 -solution with θ -pinched curvature, which we denote by $(M^\infty, g^\infty(t))$. We distinguish two cases:

Case 1: Suppose that M^∞ is compact. In this case, the diameter of $(M^{(j)}, g^{(j)}(0))$ is uniformly bounded from above by some constant which is independent of j . Hence, if j is sufficiently large, then $B^{(j)} := M^{(j)}$ is a neighborhood of the point x_j which satisfies the conclusion of Corollary 4.9. This is a contradiction.

Case 2: Suppose next that M^∞ is noncompact. If $(M^\infty, g^\infty(t))$ is locally isometric to a round cylinder, then the point x_j lies at the center of an ε -neck or an ε -quotient neck, contrary to our assumption. Thus, $(M^\infty, g^\infty(t))$ cannot be locally isometric to a round cylinder. Applying Theorem 4.7 to

$(M^\infty, g^\infty(t))$, we conclude that there exists a neighborhood $B^\infty \subset M^\infty$ of the point x_∞ which satisfies the conclusion of Corollary 4.9. Consequently, if j is sufficiently large, there exists a neighborhood $B^{(j)} \subset M^{(j)}$ of the point x_j which satisfies the conclusion of Corollary 4.9. This again contradicts our choice of x_j .

5. A CANONICAL NEIGHBORHOOD THEOREM IN HIGHER DIMENSIONS

In this section, we establish a higher-dimensional version of Perelman's Canonical Neighborhood Theorem.

Definition 5.1. Let $f : [0, \infty) \rightarrow [0, \infty)$ be a concave and increasing function such that $\lim_{s \rightarrow \infty} \frac{f(s)}{s} = \frac{1}{n-2}$. We say that a Riemannian manifold has (f, θ) -pinched curvature if the curvature tensor lies in the set

$$\begin{aligned} \{R = S + H \otimes \text{id} : S \in \text{PIC2}, \text{Ric}_0(S) = 0, \\ f(\text{tr}(H)) \text{id} - H \geq 0, \\ \text{tr}(H) - \theta \text{scal}(S) \geq 0\} \end{aligned}$$

at each point.

The following is the analogue of Perelman's Canonical Neighborhood Theorem in dimension 3:

Theorem 5.2 (cf. G. Perelman [26], Theorem 12.1). *Given a function f as above and positive numbers θ , κ , and ε , we can find a positive number r_0 such that the following holds: Let $(M, g(t))$, $t \in [0, T)$, be a compact solution to the Ricci flow which has (f, θ) -pinched curvature and is κ -noncollapsed on scales less than r_0 . Moreover, suppose that M does not contain any non-trivial incompressible space forms. Then for any point (x_0, t_0) with $t_0 \geq 1$ and $Q := \text{scal}(x_0, t_0) \geq r_0^{-2}$, the solution in $\{(x, t) : d_{g(t_0)}(x_0, x) < \varepsilon^{-\frac{1}{2}} Q^{-\frac{1}{2}}, 0 \leq t_0 - t \leq \varepsilon^{-1} Q^{-1}\}$ is, after scaling by the factor Q , ε -close to the corresponding subset of an ancient κ_0 -solution with θ -pinched curvature.*

Proof. The proof is an adaptation of the argument in Section 12.1 of [26]. We will follow the exposition in [9] and [20]. Let $C_1 = C_1(n, \theta, \varepsilon)$ denote the constant in Corollary 4.9. We define a constant $C_0 = C_0(n, \theta, \varepsilon)$ by $C_0 := 4 \max\{C_1, \varepsilon^{-1}\}$.

Suppose that the assertion is false. We can find a sequence of flows $(M^{(j)}, g^{(j)}(t))$ and a sequence of points (x_j, t_j) in space-time with the following properties:

- (i) $(M^{(j)}, g^{(j)}(t))$ does not contain any non-trivial incompressible space forms.
- (ii) $(M^{(j)}, g^{(j)}(t))$ has (f, θ) -pinched curvature.
- (iii) $(M^{(j)}, g^{(j)}(t))$ is κ -noncollapsed at all scales less than r_j .
- (iv) $t_j \geq \frac{1}{2}$ and $Q_j := \text{scal}(x_j, t_j) \geq 2^j$.

- (v) After dilating by the factor Q_j , the solution in $\{(x, t) : d_{g^{(j)}(t_j)}(x_j, x) < C_0^{\frac{1}{2}} Q_j^{-\frac{1}{2}}, 0 \leq t_j - t \leq C_0 Q_j^{-1}\}$ is not ε -close to the corresponding subset of any ancient κ_0 -solution with θ -pinched curvature.

By a standard point-picking argument, we can assume that (x_j, t_j) in addition satisfies the following condition:

- (vi) If (\tilde{x}, \tilde{t}) is a point in space-time satisfying $\text{scal}(\tilde{x}, \tilde{t}) =: \tilde{Q} > 2Q_j$ and $0 \leq t_j - \tilde{t} \leq j Q_j^{-1}$, then the solution in $\{(x, t) : d_{g^{(j)}(\tilde{t})}(\tilde{x}, x) < C_0^{\frac{1}{2}} \tilde{Q}^{-\frac{1}{2}}, 0 \leq \tilde{t} - t \leq C_0 \tilde{Q}^{-1}\}$ is, after scaling by the factor \tilde{Q} , ε -close to the corresponding subset of an ancient κ_0 -solution with θ -pinched curvature.

Our goal is to show that, if we dilate the flow $(M^{(j)}, g^{(j)}(t))$ around the point (x_j, t_j) by the factor Q_j , the rescaled flows will converge in C_{loc}^∞ to an ancient κ_0 -solution with pinched curvature. To that end, we proceed in several steps:

Step 1: The condition (vi) implies that $|DR(\tilde{x}, \tilde{t})| \leq \eta \text{scal}(\tilde{x}, \tilde{t})^{\frac{3}{2}}$ and $|D^2R(\tilde{x}, \tilde{t})| \leq \eta \text{scal}(\tilde{x}, \tilde{t})^2$ whenever $\text{scal}(\tilde{x}, \tilde{t}) > 2Q_j$ and $0 \leq t_j - \tilde{t} \leq j Q_j^{-1}$. Here, η is a large constant which is independent of j .

In particular, if (\tilde{x}, \tilde{t}) is a point in space-time satisfying $0 \leq t_j - \tilde{t} \leq \frac{1}{2} j Q_j^{-1}$, then the gradient estimate implies that $\text{scal}(x, t) \leq 4(Q_j + \text{scal}(\tilde{x}, \tilde{t}))$ for all points (x, t) satisfying $0 \leq \tilde{t} - t \leq c(Q_j + \text{scal}(\tilde{x}, \tilde{t}))^{-1}$ and $d_{g^{(j)}(\tilde{t})}(\tilde{x}, x) \leq c(Q_j + \text{scal}(\tilde{x}, \tilde{t}))^{-\frac{1}{2}}$. Here, c is a small positive constant which is independent of j .

Step 2: For each $\rho \geq 0$, let $\mathbb{M}(\rho)$ be the smallest positive number such that

$$Q_j^{-1} \text{scal}(\tilde{x}, t_j) \leq \mathbb{M}(\rho)$$

for all integers $j \in \mathbb{N}$ and all points $\tilde{x} \in M^{(j)}$ satisfying $Q_j^{\frac{1}{2}} d_{g^{(j)}(t_j)}(x_j, \tilde{x}) \leq \rho$. If no such number exists, we put $\mathbb{M}(\rho) = \infty$.

Using the local curvature bound in Step 1, we conclude that $\mathbb{M}(\rho) < \infty$ if $\rho > 0$ is sufficiently small. Let

$$\rho^* = \sup\{\rho \geq 0 : \mathbb{M}(\rho) < \infty\}.$$

We claim that $\rho^* = \infty$. We argue by contradiction, and assume that $\rho^* < \infty$.

We can find a sequence of points $\tilde{x}_j \in M^{(j)}$ such that $\limsup_{j \rightarrow \infty} Q_j^{\frac{1}{2}} d_{g^{(j)}(t_j)}(x_j, \tilde{x}_j) \leq \rho^*$ and $\liminf_{j \rightarrow \infty} Q_j^{-1} \text{scal}(\tilde{x}_j, t_j) = \infty$. Let γ_j be a minimizing geodesic in $(M^{(j)}, g^{(j)}(t_j))$ joining x_j and \tilde{x}_j , and let z_j be the point on γ closest to \tilde{x}_j with $\text{scal}(z_j, t_j) = 4Q_j$. We denote by β_j the segment of γ_j from z_j to \tilde{x}_j .

We next dilate the ball $B_{g(t_j)}(x_j, \rho^* Q_j^{-\frac{1}{2}})$ by the factor $Q_j^{\frac{1}{2}}$. After passing to a subsequence, the dilated balls converge in C_{loc}^∞ to an incomplete manifold (B^∞, g^∞) . Moreover, the geodesics γ_j and β_j converge to minimizing

geodesics γ_∞ and β_∞ in (B^∞, g^∞) . Finally, the points x_j and z_j converge to points x_∞ and z_∞ in M^∞ . Using the gradient estimate $|\nabla R| \leq \eta \text{scal}^{\frac{3}{2}}$, we conclude that the curvature of (B^∞, g^∞) must blow-up along β_∞ . Moreover, the curvature tensor of (B^∞, g^∞) lies in the cone $\mathcal{C}_{1,\theta}$.

In view of statement (vi) above, each point on β_j has a neighborhood of size $C_0 \text{scal}^{-\frac{1}{2}}$ which is ε -close to an ancient κ_0 -solution. Passing to the limit as $j \rightarrow \infty$, we conclude that each point $q \in \beta_\infty$ has a neighborhood of size $C_0 \text{scal}_{g^\infty}(q)^{-\frac{1}{2}}$ which is ε -close to an ancient κ_0 -solution. In particular, for each point $q \in \beta_\infty$, we have $C_0 \text{scal}_{g^\infty}(q)^{-\frac{1}{2}} \leq \rho^* - d_{g^\infty}(x_\infty, q)$. Moreover, each point $q \in \beta_\infty$ has a neighborhood of size $\frac{1}{2} C_0 \text{scal}_{g^\infty}(q)^{-\frac{1}{2}}$ which is ε -close to an ancient κ_0 -solution. Using Corollary 4.9 together with the fact that $C_0 \geq 4C_1$, we conclude that each point $q \in \beta_\infty$ has a canonical neighborhood B which is either a 2ε -neck; or a 2ε -cap; or a closed manifold diffeomorphic to S^n/Γ ; or a 2ε -quotient neck. Let us consider the various possibilities:

- If the canonical neighborhood of q is a closed manifold, then the curvature of (B^∞, g^∞) is bounded. This contradicts the fact that the curvature of (B^∞, g^∞) blows up along β_∞ . Therefore, this case cannot occur.
- If the canonical neighborhood of q is a quotient neck, then $M^{(j)}$ contains a quotient neck if j is sufficiently large. Theorem A.1 then implies that $M^{(j)}$ contains a non-trivial incompressible space-form for j sufficiently large, contrary to our assumption. Hence, this case cannot occur.
- Finally, if $\rho^* - d_{g^\infty}(x_\infty, q)$ is sufficiently small and the canonical neighborhood of q is a 2ε -cap, then β_∞ must enter and exit this cap, but this is impossible since β_∞ is a minimizing geodesic. Consequently, this case cannot occur if $\rho^* - d_{g^\infty}(x_\infty, q)$ is sufficiently small.

To summarize, if $q \in \beta_\infty$ and $\rho^* - d_{g^\infty}(x_\infty, q)$ is sufficiently small, then q has a canonical neighborhood which is a 2ε -neck. Let U denote the union of the canonical neighborhoods of all points $q \in \beta_\infty$, where $\rho^* - d_{g^\infty}(x_\infty, q)$ is sufficiently small.

By work of Hamilton [17], U admits a foliation by constant mean curvature sphere Σ_s . We parametrize the surfaces Σ_s so that the surfaces Σ_s move outward as $s \searrow 0$, and the lapse function $v : \Sigma_s \rightarrow \mathbb{R}$ has mean value 1 for each $s > 0$. Note that $\frac{1}{2} \leq v \leq 2$ on each leaf Σ_s . In particular, $\rho^* - d_{g^\infty}(x_\infty, q)$ is comparable to s for each point $q \in \Sigma_s$.

Let $H(s)$ denote the mean curvature of Σ_s . Then

$$-H'(s) = \Delta_{\Sigma_s} v + |A|^2 v + \text{Ric}_{g^\infty}(\nu, \nu) v \geq \Delta_{\Sigma_s} v + \frac{1}{n-1} H(s)^2 v.$$

We now take the mean value over Σ_s . Using the fact that v has mean 1 and $\Delta_{\Sigma_s} v$ has mean 0, we obtain

$$-H'(s) \geq \frac{1}{n-1} H(s)^2.$$

From this, we deduce that

$$H(s) \leq \frac{n-1}{s}.$$

Using again the fact that v has mean 1, we obtain

$$\frac{d}{ds} \text{area}_{g^\infty}(\Sigma_s) = H(s) \int_{\Sigma_s} v = H(s) \text{area}_{g^\infty}(\Sigma_s) \leq \frac{n-1}{s} \text{area}_{g^\infty}(\Sigma_s).$$

Consequently, the function $s^{1-n} \text{area}_{g^\infty}(\Sigma_s)$ is monotone decreasing. In particular,

$$\liminf_{s \rightarrow 0} s^{1-n} \text{area}_{g^\infty}(\Sigma_s) > 0$$

or, equivalently,

$$\limsup_{s \rightarrow 0} \sup_{q \in \Sigma_s} s^{-2} \text{scal}_{g^\infty}(q) < \infty.$$

On the other hand, since $C_0 \text{scal}_{g^\infty}(q)^{-\frac{1}{2}} \leq \rho^* - d_{g^\infty}(x_\infty, q)$ for each point $q \in \beta_\infty$, we know that

$$\liminf_{s \rightarrow 0} \sup_{q \in \Sigma_s} s^{-2} \text{scal}_{g^\infty}(q) > 0$$

or, equivalently,

$$\limsup_{s \rightarrow 0} s^{1-n} \text{area}_{g^\infty}(\Sigma_s) < \infty.$$

We now dilate the manifold U around a point on Σ_s by the factor s^{-1} , where $s > 0$ is small. In the limit, we obtain a smooth metric which is a piece of a cone. Using the local curvature estimate in Step 1, we can locally extend the metric backwards in time to a solution of the Ricci flow. To summarize, we obtain a (locally defined) solution to the Ricci flow whose curvature tensor lies in $\mathcal{C}_{1,\theta}$ and which, at the final time, is a piece of a cone. This contradicts Proposition 3.1. Thus, $\rho^* = \infty$.

Step 3: We now dilate the manifold $(M^{(j)}, g^{(j)}(t_j))$ around the point x_j by the factor Q_j . Using the curvature bounds established in Steps 1 and 2 together with the κ -noncollapsing condition, we conclude that, after passing to a subsequence, the rescaled manifolds converge in the Cheeger-Gromov sense to a smooth limit manifold, which we denote by (M^∞, g^∞) . The condition (vi) implies that $|DR(\tilde{x})| \leq \eta \text{scal}(\tilde{x})^{\frac{3}{2}}$ and $|D^2 R(\tilde{x})| \leq \eta \text{scal}(\tilde{x})^2$ for all points $\tilde{x} \in M^\infty$ satisfying $\text{scal}(\tilde{x}) \geq 4$. Moreover, the curvature tensor of (M^∞, g^∞) lies in the cone $\mathcal{C}_{1,\theta}$.

We claim that the limit (M^∞, g^∞) has bounded curvature. If the curvature tensor of (M^∞, g^∞) lies in the interior of the cone $\mathcal{C}_{1,\theta}$ at all points, then this follows directly from Proposition 3.3. Hence, it remains to consider the case when the curvature tensor of (M^∞, g^∞) lies on the boundary of the

cone $\mathcal{C}_{1,\theta}$ at some point. Using the local curvature bound in Step 1 we can locally extend the metric g^∞ backwards in time to a solution of the Ricci flow. Proposition 3.1 then implies that (M^∞, g^∞) locally splits as a product. This easily implies that (M^∞, g^∞) has bounded curvature. This proves the claim.

Step 4: We next show that (M^∞, g^∞) can be extended backwards in time to an ancient solution. For each $\tau \geq 0$, let $\mathbb{L}(\tau)$ be the smallest positive number such that

$$\limsup_{j \rightarrow \infty} Q_j^{-1} \text{scal}(\tilde{x}_j, \tilde{t}_j) \leq \mathbb{L}(\tau)$$

for every sequence of points $(\tilde{x}_j, \tilde{t}_j)$ in space-time satisfying $0 \leq t_j - \tilde{t}_j \leq \tau Q_j^{-1}$ and $\limsup_{j \rightarrow \infty} Q_j^{\frac{1}{2}} d_{g^{(j)}(\tilde{t}_j)}(x_j, \tilde{x}_j) < \infty$. If no such number exists, we put $\mathbb{L}(\tau) = \infty$.

We have shown in Step 3 that the curvature of (M^∞, g^∞) is bounded from above by some constant $\Lambda \geq 1$. This implies $\mathbb{L}(0) \leq \Lambda$. Using the local curvature bound in Step 1, we conclude that $\mathbb{L}(\tau) \leq 8(\Lambda + 1)$ if $\tau > 0$ is sufficiently small. Let

$$\tau^* = \sup\{\tau \geq 0 : \mathbb{L}(\tau) < \infty\}.$$

We first show that $\sup_{\tau \in [0, \tau^*)} \mathbb{L}(\tau) \leq \Lambda$. Indeed, if $\sup_{\tau \in [0, \tau^*)} \mathbb{L}(\tau) > \Lambda$, then we dilate the flow $(M^{(j)}, g^{(j)}(t))$ around the point (x_j, t_j) by the factor Q_j . Using the fact that $\mathbb{L}(\tau) < \infty$ for each $\tau \in [0, \tau^*)$ together with the κ -noncollapsing condition, we conclude that the rescaled flows converge in the Cheeger-Gromov sense to a solution $(M^\infty, g^\infty(t))$ which is defined for $t \in (-\tau^*, 0]$ and has bounded curvature for each t . Moreover, the curvature tensor of the limit flow $(M^\infty, g^\infty(t))$ lies in the cone $\mathcal{C}_{1,\theta}$. Using the Harnack inequality, we conclude that the curvature of the limit flow $(M^\infty, g^\infty(t))$ is bounded from above by Λ . This implies $\sup_{\tau \in [0, \tau^*)} \mathbb{L}(\tau) \leq \Lambda$.

We claim that $\tau^* = \infty$. To prove this, we consider an arbitrary time $\hat{\tau} \in (0, \tau^*)$, and put $\hat{t}_j := t_j - \hat{\tau} Q_j^{-1}$. Using the local curvature estimate in Step 1 and the fact that $\mathbb{L}(\hat{\tau}) \leq \Lambda$, we conclude that there exists a positive number δ (independent of j and $\hat{\tau}$) with the property that

$$\limsup_{j \rightarrow \infty} Q_j^{-1} \text{scal}(\tilde{x}_j, \tilde{t}_j) \leq 8(\Lambda + 1)$$

for every sequence of points $(\tilde{x}_j, \tilde{t}_j)$ in space-time satisfying $0 \leq \hat{t}_j - \tilde{t}_j \leq \delta Q_j^{-1}$ and $\limsup_{j \rightarrow \infty} Q_j^{\frac{1}{2}} d_{g^{(j)}(\tilde{t}_j)}(x_j, \tilde{x}_j) < \infty$. This implies

$$\limsup_{j \rightarrow \infty} Q_j^{-1} \text{scal}(\tilde{x}_j, \tilde{t}_j) \leq 8(\Lambda + 1)$$

for every sequence of points $(\tilde{x}_j, \tilde{t}_j)$ in space-time satisfying $0 \leq \hat{t}_j - \tilde{t}_j \leq \delta Q_j^{-1}$ and $\limsup_{j \rightarrow \infty} Q_j^{\frac{1}{2}} d_{g^{(j)}(\tilde{t}_j)}(x_j, \tilde{x}_j) < \infty$. Thus, $\mathbb{L}(\hat{\tau} + \delta) \leq 8(\Lambda + 1)$. In particular, $\hat{\tau} + \delta \leq \tau^*$ by definition of τ^* . Since $\hat{\tau} \in (0, \tau^*)$ is arbitrary

and $\delta > 0$ is independent of $\hat{\tau}$, we obtain a contradiction. Thus, $\tau^* = \infty$.

Step 5: Finally, we dilate the flow $(M^{(j)}, g^{(j)}(t))$ around the point (x_j, t_j) by the factor Q_j . Using the fact that $\sup_{\tau \in [0, \infty)} \mathbb{L}(\tau) \leq \Lambda$ together with the κ -noncollapsing condition, we conclude that the rescaled flows converge in the Cheeger-Gromov sense to an ancient solution which has bounded curvature and is κ -noncollapsed. Moreover, the curvature tensor of the limit flow lies in the cone $\mathcal{C}_{1, \theta}$. Thus, the limit flow is an ancient κ -solution with θ -pinched curvature. By Theorem 4.8, the limit flow is κ_0 -noncollapsed. This contradicts statement (v) above. This completes the proof.

Corollary 5.3 (cf. G. Perelman [26], Theorem 12.1). *Given a function f as above and positive numbers θ , κ , and ε , we can find a positive number r_0 such that the following holds: Let $(M, g(t))$, $t \in [0, T)$, be a compact solution to the Ricci flow which has (f, θ) -pinched curvature and is κ -noncollapsed on scales less than r_0 . Moreover, suppose that M does not contain any non-trivial incompressible space forms. Then for any point (x_0, t_0) with $t_0 \geq 1$ and $Q := \text{scal}(x_0, t_0) \geq r_0^{-2}$, there exists a neighborhood B of x_0 such that $B_{g(t_0)}(x_0, (2C_1)^{-1} \text{scal}(x_0, t_0)^{-\frac{1}{2}}) \subset B \subset B_{g(t_0)}(x_0, 2C_1 \text{scal}(x_0, t_0)^{-\frac{1}{2}})$ and $(2C_2)^{-1} \text{scal}(x_0, t_0) \leq \text{scal}(x, t_0) \leq 2C_2 \text{scal}(x_0, t_0)$ for all $x \in B$. Finally, B falls into one of the following three categories:*

- B is a 2ε -neck.
- B is a 2ε -cap in the sense that B is diffeomorphic to a ball and the boundary ∂B is the cross-sectional sphere of a 2ε -neck.
- B is a closed manifold diffeomorphic to S^n/Γ .

Here, $C_1 = C_1(n, \theta, \varepsilon)$ and $C_2 = C_2(n, \theta, \varepsilon)$ are the constants appearing in Corollary 4.9.

6. THE BEHAVIOR OF THE FLOW AT THE FIRST SINGULAR TIME

As in [27], we are now able to give a precise description of the behavior of the flow at the first singular time. Let us fix a compact initial manifold (M, g_0) with the property that the curvature tensor of (M, g_0) lies in the interior of the cone $\mathcal{C}_{2,0}$, and let $(M, g(t))$, $t \in [0, T)$, denote the unique maximal solution to the Ricci flow with initial metric g_0 . By Theorem 2.6, there exists a pair (f, θ) such that $(M, g(t))$ has (f, θ) -pinched curvature for all $t \in [0, T)$. Moreover, by work of Perelman [26], there exists a constant $\kappa > 0$ such that $(M, g(t))$ is κ -noncollapsed for all $t \in [0, T)$. Thus, the Canonical Neighborhood Theorem can be applied. In particular, we have $|DR| \leq \eta \text{scal}^{\frac{3}{2}}$ and $|D^2R| \leq \eta \text{scal}^2$ at all points where the scalar curvature is sufficiently large. This implies that the set

$$\Omega := \{x \in M : \limsup_{t \rightarrow T} \text{scal}(x, t) < \infty\}$$

is open. Moreover, if $\rho > 0$ is sufficiently small, then the set

$$\Omega_\rho := \{x \in M : \limsup_{t \rightarrow T} \text{scal}(x, t) < \rho^{-2}\}$$

is open as well. The metrics $g(t)$ converge smoothly to a smooth metric $g(T)$ which is defined on Ω .

By the Canonical Neighborhood Theorem, each point in $\Omega \setminus \Omega_\rho$ is contained in one of the following sets:

- an ε -tube with boundary components in Ω_ρ
- an ε -cap with boundary in Ω_ρ
- an ε -horn with boundary in Ω_ρ
- a double ε -horn
- a capped ε -horn
- a closed manifold diffeomorphic to S^n/Γ

As in Perelman's paper [27], we perform a cutoff surgery on every ε -horn with boundary in Ω_ρ . Moreover, we remove all double ε -horns, all capped ε -horns, and all closed manifolds diffeomorphic to S^n/Γ . (The ε -tubes and ε -caps with boundary in Ω_ρ are left unchanged.)

Proposition 6.1. *The pre-surgery manifold is a connected sum of the post-surgery manifold together with finitely many additional pieces. Each additional piece is diffeomorphic to a quotient of S^n or to a compact quotient of $S^{n-1} \times \mathbb{R}$.*

The following result can be viewed as the higher dimensional analogue of a theorem of R. Hamilton (cf. [17], Theorem D3.1):

Proposition 6.2. *If $(\Omega, g(T))$ has (f, θ) -pinched curvature, then this remains so after surgery.*

Proof. Let us consider a neck of the form $S^{n-1} \times [-10, 10]$, and let g denote the Riemannian metric on the neck. In other words, g is very close to the standard metric \bar{g} on $S^{n-1} \times [-10, 10]$ after rescaling. By assumption, the Riemann curvature tensor of g can be written as $R = S + H \otimes \text{id}$, where $S \in \text{PIC2}$, $\text{Ric}_0(S) = 0$, $f(\text{tr}(H)) \text{id} - H \geq 0$, and $\text{tr}(H) - \theta \text{scal}(S) \geq 0$.

As in [17], the surgically modified metric is given by $\tilde{g} = e^{-2\varphi} g$, where $\varphi = e^{-\frac{1}{z}}$ and z denotes the height function on $S^{n-1} \times [-10, 10]$. Computing in an orthonormal frame, the standard formula for the change of the curvature tensor under a conformal change of the metric takes the form

$$\tilde{R} = e^{2\varphi} R + e^{2\varphi} \left(D^2\varphi + d\varphi \otimes d\varphi - \frac{1}{2} |d\varphi|^2 \text{id} \right) \otimes \text{id}.$$

Hence, we may write $\tilde{R} = \tilde{S} + \tilde{H} \otimes \text{id}$, where

$$\tilde{S} = e^{2\varphi} S$$

and

$$\tilde{H} = e^{2\varphi} H + e^{2\varphi} \left(D^2\varphi + d\varphi \otimes d\varphi - \frac{1}{2} |d\varphi|^2 \text{id} \right).$$

If $z > 0$ is sufficiently small, we have

$$\mathrm{tr}(\tilde{H}) - \mathrm{tr}(H) \geq (1 - c) z^{-4} e^{-\frac{1}{z}},$$

where c is a positive constant that can be made arbitrarily small. Similarly, we have

$$\lambda_n(\tilde{H}) - \lambda_n(H) \leq c z^{-4} e^{-\frac{1}{z}},$$

where again c is a positive constant that can be made arbitrarily small. Hence, if $z > 0$ is sufficiently small, then we obtain

$$\frac{1}{n-2} (\mathrm{tr}(\tilde{H}) - \mathrm{tr}(H)) - (\lambda_n(\tilde{H}) - \lambda_n(H)) \geq \left(\frac{1-c}{n-2} - c \right) z^{-4} e^{-\frac{1}{z}} > 0.$$

Since $f' \geq \frac{1}{n-2}$, it follows that

$$\begin{aligned} f(\mathrm{tr}(\tilde{H})) - \lambda_n(\tilde{H}) &\geq f(\mathrm{tr}(H)) - \lambda_n(H) \\ &\quad + \frac{1}{n-2} (\mathrm{tr}(\tilde{H}) - \mathrm{tr}(H)) - (\lambda_n(\tilde{H}) - \lambda_n(H)) \\ &> 0 \end{aligned}$$

if $z > 0$ is sufficiently small. Thus, $f(\mathrm{tr}(\tilde{H})) \mathrm{id} - \tilde{H} \geq 0$ if $z > 0$ is sufficiently small. Moreover, since $\mathrm{tr}(\tilde{H}) \geq e^{2\varphi} \mathrm{tr}(H)$, we obtain

$$\mathrm{tr}(\tilde{H}) - \theta \mathrm{scal}(\tilde{S}) \geq e^{2\varphi} (\mathrm{tr}(H) - \theta \mathrm{scal}(S)) \geq 0$$

if $z > 0$ is sufficiently small. This shows that the surgically modified metric has (f, θ) -pinched curvature when $z > 0$ is sufficiently small. Finally, if z is bounded away from 0, then it is easy to see that the curvature tensor of the surgically modified metric lies in the cone $\mathcal{C}_{1,\theta}$. Therefore, the surgically modified metric has (f, θ) -pinched curvature everywhere.

We now consider the surgically modified manifold, and evolve it again by the Ricci flow. As explained in Section 5 of Perelman's paper [27], we can choose the surgery cutoff in such a way that the κ -noncollapsing property and the Canonical Neighborhood Theorem hold for the surgically modified flow. From this, Theorem 1.1 follows.

APPENDIX A. A HIGHER-DIMENSIONAL ANALOGUE OF THEOREM C4.1 IN HAMILTON'S PAPER [17]

Theorem A.1 (cf. R. Hamilton [17], Theorem C4.1). *Let M be a compact manifold with positive isotropic curvature. Moreover, let us consider a noncompact quotient of the round cylinder of the form $(S^{n-1} \times \mathbb{R})/\Gamma$. Suppose that $(S^{n-1} \times [-L, L])/\Gamma$ can be embedded into M in such a way that the induced metric is sufficiently close to the standard metric on $(S^{n-1} \times [-L, L])/\Gamma$ in the C^4 -norm. Then $\pi_1((S^{n-1} \times \{0\})/\Gamma) \rightarrow \pi_1(M)$ injects.*

Proof. Let Γ_1 denote the image of Γ under the canonical projection from $\mathrm{Isom}(S^{n-1} \times \mathbb{R})$ to $\mathrm{Isom}(\mathbb{R})$. Since the quotient $(S^{n-1} \times \mathbb{R})/\Gamma$ is noncompact, the group Γ_1 is either trivial, or else it has order 2 and is generated by the

reflection $s \rightarrow -s$. In either case, the center sphere $S^{n-1} \times \{0\}$ is invariant under the action of Γ .

If $\pi_1((S^{n-1} \times \{0\})/\Gamma) \rightarrow \pi_1(M)$ fails to be injective, we can find a non-trivial loop $\alpha : S^1 \rightarrow (S^{n-1} \times \{0\})/\Gamma \subset M$ which bounds an immersed two-dimensional disk in M . In other words, there exists a smooth map $f : B^2 \rightarrow M$ such that $f|_{\partial B^2} = \alpha$. Following [17], we modify the Riemannian metric on M slightly so that the center slice $(S^{n-1} \times \{0\})/\Gamma$ is totally geodesic. Since the cylinder has strictly positive isotropic curvature, this can be done in such a way that M still has positive isotropic curvature. We now minimize energy among all two-dimensional disks $\tilde{f} : B^2 \rightarrow M$ with the property that $\tilde{f}(\partial B^2)$ is contained in the center slice $(S^{n-1} \times \{0\})/\Gamma$ and \tilde{f} and f represent the same homotopy class. This gives a free-boundary harmonic disk of index 0, contradicting Theorem 2.8 in [11].

We note that the proof of Theorem A.1 carries over to the noncompact setting; this is discussed in detail in Chen-Zhu [9], Lemma 3.1.

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